CALCULUS 30: UNIT 3: DAY 7 – IMPLICIT DIFFERENTIATION (SECTION 2.7)

To learn and apply implicit differentiation versus explicit differentiation.

- So far, our functions have been **explicitly defined**, which is when y is already isolated.
  - Ex: $y = x^5 + 3x - 1$
- We will now be working with **implicitly defined** functions, where we cannot solve for y.
  - Ex: $x^2 - y^3 + 3xy = 1$

Review of Notations:

Using the chain Rule multiply the coefficient by the exponent and subtract 1 from the exponent then multiply by the derivative of the variable which is $\frac{dx}{dx}$

$\frac{d}{dx}(x^3)$ means: find the derivative of x^3 with respect to x. $\frac{d}{dx}(x^3) = 3x^2 \cdot \frac{dx}{dx}$ note: This is equal to 1

$\frac{d}{dx}(y^4)$ means: find the derivative of y^4 with respect to x. $\frac{d}{dx}(y^4) =$ This is challenging as there are no x's

- To take the derivative of a function that is defined implicity, we take the derivaitive from left to right, and wherever there is a value of *y* in the equation, we need to use the chain rule and multiply that term by $\frac{dy}{dx}$. We normally use the chain rule when taking the derivative of x values but the chain rule of those terms ends up being $\frac{dx}{dx}$ which reduces to 1.

notice there is no equation

Ex #1: a) Find $\frac{d}{dx}(y^2)$. *use chain rule*

This question is read as, "Find the derivative of y^2 with respect to x"

$$= 2y \cdot \frac{dy}{dx}$$

b) Find $\frac{d}{dx}(y^2)$, if $y = (x^2 + 4x + 3)$

substitute

$$\frac{d}{dx}(x^2+4x+3)^2$$

$$= 2\underbrace{(x^2+4x+3)}_{\text{equal to } y} \cdot (2x+4)$$

This is the derivative of y $\therefore \frac{dy}{dx}$

If given a question that is an equation for example $y = 3(x^2+1)^4$
Take the derivative of both sides.

derivative of "y" with respect to x.

$$y = 3(x^2+1)^4$$

use chain rule to find derivative with respect to "x"

$$\frac{dy}{dx} = 12(x^2+1)^3(2x) \cdot \frac{dx}{dx}$$

you can write this but it is equal to 1.

$$\boxed{\frac{dy}{dx} = 12(x^2+1)^3(2x)}$$

Note: These are finding derivatives of expressions.

Ex#1 c) : Differentiate from left to right with respect to x :

"Product rule"

a) $\frac{d}{dx}(9x^2 - 4y^{-\frac{1}{4}})$

$= 18x \cdot \frac{dx}{dx} + y^{-\frac{5}{4}} \cdot \frac{dy}{dx}$

derivative of "x" with respect to "x", equals 1

$= 18x + \frac{1}{y^{\frac{5}{4}}} \cdot \frac{dy}{dx}$

$= 18x + \frac{1}{\sqrt[4]{y^5}} \cdot \frac{dy}{dx}$

$= 18x + \frac{1}{y\sqrt[4]{y}} \cdot \frac{dy}{dx}$

Simplifying so you have no negative or rational exponents

b) $\frac{d}{dx}(2x^3y^4)$

$= 6x^2y^4 + \underbrace{2x^3 \cdot 4y^3}_{\text{multiply what you can}} \cdot \frac{dy}{dx}$

$= 6x^2y^4 + 8x^3y^3\frac{dy}{dx}$

USING IMPLICIT DIFFERENTIATION when working with EQUATIONS containing a mixture of x and y:

STEP 1: Differentiate **both sides of the equation**, from left to right, **with respect to x**.

STEP 2: Collect all the terms with $\frac{dy}{dx}$ on one side of the equation

STEP 3: Factor out the $\frac{dy}{dx}$

STEP 4: Isolate $\frac{dy}{dx}$.

Using **implicit differentiation is easier than explicit differentiation**. Here is an example as to why:

Ex #2: If $x^2 + y^2 = 169$, find $\frac{dy}{dx}$ (both explicitly and implicitly).

Unit 3: Day 7 - Implicit Differentiation.

Ex#2) If $x^2+y^2=169$, find $\frac{dy}{dx}$. (both explicitly and implicitly)

Note: To differentiate explicitly isolate the "y" first. (We have been using this method already because all equations had y-isolated already

Method 1: **Using Explicit Differentiation**

Step 1: Isolate the y-variable in the equation

$$x^2 - x^2 + y^2 = 169 - x^2$$

$$\sqrt{y^2} = \sqrt{169 - x^2}$$

$$y = \sqrt{169 - x^2}$$

$$y = (169 - x^2)^{1/2}$$ ← re-write with rational exponent

Step 2: Take the derivative of each side of the equal sign with respect to "x".

Use chain rule to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}(169 - x^2)^{-1/2}(-2x)\cdot\frac{dx}{dx}$$ ← **note:** $\frac{dx}{dx} = 1$

$$\frac{dy}{dx} = \frac{-2x}{2}(169 - x^2)^{-1/2}$$ * multipy what you can, and simplify *

$$\boxed{\frac{dy}{dx} = \frac{-x}{\sqrt{169 - x^2}}}$$

Notice: This is your original function $y = \sqrt{169 - x^2}$

Method 2: **Implicit Differentiation**

Step 1: Differentiate left to right with respect to "x"

$$x^2 + y^2 = 169$$

$$2x\cdot\frac{dx}{dx} + 2y\cdot\frac{dy}{dx} = 0$$

Step 2: Isolate $\frac{dy}{dx}$

$$2x - 2x + 2y\cdot\frac{dy}{dx} = 0 - 2x$$

$$\frac{2y}{2y}\cdot\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Notice: Our denominator is y, thinking back to our original function $y = \sqrt{169 - x^2}$ (once we isolated the "y") we could substitute this in for "y" to get the same answer as method 1.

$$\boxed{\frac{dy}{dx} = \frac{-x}{\sqrt{169 - x^2}}}$$

- NO Fractional Coefficients
- "A" is positive

$AX+BY+C=0$

Ex #3: Suppose $x^2y+2y^2-x=3$. using implicit.

a) Find $\frac{dy}{dx}$

b) Find the equation of the tangent line at (1,2) in standard form

a)

$\boxed{\text{Step 1}}$: Take the derivate of each term Left to Right

$x^2y + 2y^2 - x = 3$

use product Rule   use chain rule

Remember: The derivative of "y" with respect to "x" is $\frac{dy}{dx}$

$2xy + x^2 \cdot \frac{dy}{dx} + 4y \cdot \frac{dy}{dx} - 1 = 0$ $\quad -2xy \quad +1$

$\boxed{\text{Step 2}}$: Isolate $\frac{dy}{dx}$

$x^2 \cdot \frac{dy}{dx} + 4y\frac{dy}{dx} = -2xy + 1$ → Factor out a GCF of $\frac{dy}{dx}$ out of both terms on the left side

$\frac{dy}{dx}(x^2+4y) = -2xy+1$

$\frac{\frac{dy}{dx}(x^2+4y)}{(x^2+4y)} = \frac{-2xy+1}{(x^2+4y)}$

$\boxed{\frac{dy}{dx} = \frac{-2xy+1}{x^2+4y}}$ ← Remember we can use this to find the slope of the tangent line at any point (x,y) on $x^2y+2y^2-x=3$

b) $x=1 \quad y=2$

$m = \frac{dy}{dx} = \frac{-2(1)(2)+1}{(1)^2+4(2)}$

$m = \frac{-4+1}{1+8}$

$m = \frac{-3 \div 3}{9 \div 3}$

$m = -\frac{1}{3}$ ∴ The slope is $-\frac{1}{3}$

Equation using $m=-\frac{1}{3}$ and point $(1,2)$

$y-2 = -\frac{1}{3}(x-1)$ ← Convert to Standard Form

$(3)y - 2(3) = -\frac{1}{3}x(3) + \frac{1}{3}(3)$ ← Multiply by the LCD "Lowest Common Denominator"

$3y - 6 = -x + 1$ $\quad -1 \quad +x$ ← Set equal to 0. Make sure your "x" coefficient is positive

$\boxed{x+3y-7=0}$

The equation of the tangent line at (1,2) on $x^2y+2y^2-x=3$ is $x+3y-7=0$.

Unit 3: DAY 7 ASSIGNMENT :

TEXTBOOK P 107 #1, 2a-d, f, 3, 5a(Leave answer in standard form)
Challenge yourself with P107 # 6, 7

Unit 3: Day 7 Implicit Differentiation

Ex #4/ a) Find y' if $x^2+\sqrt{y}=x^2y^3+5$ b) Find the slope of the tangent line at $(-1,1)$ "add this to the question"

Step 1: Take the derivative of each term from left to right

$$\underbrace{x^2+y^{1/2}=x^2y^3}_{\text{use product rule}}+5$$

Note: $\frac{dy}{dx}=y'$

$$2x+\frac{1}{2}y^{-1/2}\cdot y'=2xy^3+x^2 3y^2\cdot y'+0$$

Step 2: Isolate y'

$$2x+\frac{1}{2}y^{-1/2}y'=2xy^3+3x^2y^2y'$$

$-2x$ $\quad -3x^2y^2y'$ $\quad -3x^2y^2y'$ $\quad -2x$

$$\underbrace{\frac{1}{2}y^{-1/2}y'-3x^2y^2y'}_{\text{Factor out a GCF of } y' \text{ from both terms}}=2xy^3-2x$$

$$\frac{y'\left(\frac{1}{2}y^{-\frac{1}{2}}-3x^2y^2\right)}{\left(\frac{1}{2}y^{-1/2}-3x^2y^2\right)}=\frac{2xy^3-2x}{\frac{1}{2}y^{-\frac{1}{2}}-3x^2y^2}$$

$$\boxed{y'=\frac{2xy^3-2x}{\frac{1}{2\sqrt{y}}-3x^2y^2}}$$

Note: We are using y' to find the slope of the tangent line no need to simplify here by finding a common denominator

b) Find the slope of the tangent line at $(-1,1)$ ★substitute $x=-1$ and $y=1$ into y'★

$$m=\frac{2(-1)(1)^3-2(-1)}{\frac{1}{2\sqrt{1}}-3(-1)^2(1)^2}$$

$$m=\frac{-2(1)+2}{\frac{1}{2}-3}$$

$$m=\frac{-2+2}{\frac{1}{2}-3}$$

$$m=\frac{0}{\frac{1}{2}-3}$$

$$\boxed{m=0}$$

To learn and apply higher order derivatives.

Higher Order Derivatives:

- We can take the derivative of a derivative function, and the derivative of that function and so on.
- A first derivative is written as $f'(x)$ or $\frac{dy}{dx}$ or y'
  - A first derivative represents the slope of a tangent line or rate of change (how the slope of the original function changes). A common example of the first derivative is that velocity is a first derivative of a distance function.
- A second derivative is written as $f''(x)$ or $\frac{d^2y}{dx^2}$ or y''
  - A second derivative measures how fast the first derivative function (often velocity) is changing, specifically how the rate of change/slope of the tangent line of the original function changes. A common example of the second derivative is acceleration in that acceleration is the second derivative of a distance function (but the first derivative of a velocity function)
- A third derivative is written as $f'''(x)$ or $\frac{d^3y}{dx^3}$. or y'''
  - An example of a third derivative measures how fast acceleration is changing with respect to time. In physics this can also be known as jerk/jolt/surge or lurch.
- If a distance formula y = s(t), then
  y' = v(t) and
  y'' = a(t).
  - If, however, the initial function y = v(t) then it's first derivative y' = a(t)

Ex #1: Find $\frac{d^2y}{dx^2}$ if $y = x^6$ This is asking you to find the 2nd derivative

$\boxed{\text{Step 1:}}$ Take the derivative of bothsides

$$y = x^6$$

$$\frac{dy}{dx} = 6x^5$$

step 2: Take the derivative of the first derivative

$$\frac{d^2y}{d^2x} = 30x^4$$

Ex #2: Find the second derivative of $f(x) = 5x^2 + \sqrt{x}$

$f(x) = 5x^2 + x^{1/2}$

$f'(x) = 10x + \frac{1}{2}x^{-1/2}$

$f''(x) = 10 - \frac{1}{4}x^{-3/2}$

Simplify

$f''(x) = \frac{10 \cdot 4\sqrt{x^3}}{1 \cdot 4\sqrt{x^3}} - \frac{1}{4\sqrt{x^3}}$ ← Find common denominator

$f''(x) = \frac{40\sqrt{x^3} - 1}{4\sqrt{x^3}}$ ← simplify radicals

$\boxed{f''(x) = \frac{40x\sqrt{x} - 1}{4x\sqrt{x}}}$

Ex #3: Find $f''(1)$ if $f(x) = (2 - x^2)^{10}$ "This is asking you to find the second derivative when $x=1$"

$f'(x) = 10(2-x^2)^9(-2x)$ ← use chain rule to find first derivative "

$f'(x) = -20x(2-x^2)^9$ ← multiply what you can

$f''(x) = \underbrace{-20x\left(9(2-x^2)^8(-2x)\right)}_{\text{use product rule with chain rule}} + -20(2-x^2)^9$ ← Find the 2nd derivative

$f''(x) = (-2x)(-20x)(9)(2-x^2)^8 - 20(2-x^2)^9$ ← multiply what you can in each term to simplify

$f''(x) = 360x(2-x^2)^8 - 20(2-x^2)^9$ ← If you were asked to find $f''(x)$ you could factor out a GCF here. Since we are not asked for $f''(x)$, we are asked for $f''(1)$, substitute $x=1$ and solve

$f''(1) = 360(1)(2-(1)^2)^8 - 20(2-(1)^2)^9$

$f''(1) = 360(1) - 20$

$\boxed{f''(1) = 340}$

Ex #4: If $x^3 + y^3 = 5$, use implicit differentiation to find $\frac{d^2y}{dx^2}$. } 2nd derivative

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 0$$

$$-3x^2$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 0 - 3x^2$$

$$\frac{3y^2}{3y^2} \frac{dy}{dx} = \frac{-3x^2}{3y^2}$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$$

$$\frac{dy}{dx} = \frac{-x^2}{y^2}$$

use quotient rule to find 2nd derivative

$$\frac{d^2y}{d^2x} = \frac{y^2(-2x) - (-x^2)(2y \cdot \frac{dy}{dx})}{[y^2]^2}$$

$$\frac{d^2y}{d^2x} = \frac{-2xy^2 + 2x^2y \cdot \frac{dy}{dx}}{y^4}$$

$\frac{dy}{dx} = \frac{-x^2}{y^2}$ substitute in

$$\frac{d^2y}{d^2x} = \left[-2xy^2 + 2x^2y\left(\frac{-x^2}{y^2}\right)\right] \div y^4$$

$$\frac{d^2y}{d^2x} = \left[-2xy^2 - \frac{2x^4y}{y^2}\right] \cdot \frac{1}{y^4}$$

$$\frac{d^2y}{d^2x} = -\frac{2xy^2 \cdot y^2}{y^4 \cdot y^2} - \frac{2x^4y}{y^6}$$ ← common denominator

$$\frac{d^2y}{d^2x} = \frac{-2xy^4 - 2x^4y}{y^6}$$ ← factor out "y" in numerator

$$\frac{d^2y}{d^2x} = \frac{y(-2x^3 - 2x^4)}{y^{6\,5}}$$ ← reduce "y"

$$\boxed{\frac{d^2y}{d^2x} = \frac{-2x^3 - 2x^4}{y^5}}$$

Unit 3 DAY 8 ASSIGNMENT :

TEXTBOOK P 111 #1odd, 2, 3, 4, 5, 7

Unit 3 REVIEW ASSIGNMENT

P 112 #4a-n, 5a, 7abc, 8, 9ade PLUS the following:

1. **Find the coordinates of two points on the graph of $f(x) = 4x^3 + x^2 + 2x + 8$ at which the slope of the tangent line is 4.**
2. **Find $\frac{d^2y}{dx^2}$ given the equation $2y^2 - xy = 6$**

Solutions: 1. $\left(-\frac{1}{2}, \frac{3}{4}\right)$ and $\left(\frac{1}{3}, \frac{44}{27}\right)$ 2. $\frac{d^2y}{dx^2} = \frac{12}{(4y-x)^3}$