To learn and apply the power rule and the sum and difference rules for differentiation..

Do you see any patterns in the following questions between f(x) and the FINAL answer for f'(x)? Can we use this pattern to jump right to the answer without doing any of the "limit" work in between?

If $f(x) = 3x^2$, find $f`(x)$

$$f'(x)=\lim_{h\to 0}\frac{\left[3(x+h)^2\right]-\left[3x^2\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{\left[3x^2+6xh+3h^2\right]-\left[3x^2\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{\left[3x^2+6xh+3h^2\right]-\left[3x^2\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{\left[3x^2+6xh+3h^2\right]-\left[3x^2\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{3h^2+6xh}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{h(3h+6x)}{h}$$
$$f'(x)=\lim_{h\to 0}3h+6x$$
$$f`(x)=6x$$

If $f(x) = 5x^3$, find $f`(x)$

$$f'(x)=\lim_{h\to 0}\frac{\left[5(x+h)^3\right]-\left[5x^3\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{\left[5x^3+15x^2h+15xh^2+5h^3\right]-\left[5x^3\right]}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{15x^2h+15xh^2+5h^3}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{h(15x^2+15xh+5h^2)}{h}$$
$$f'(x)=\lim_{h\to 0}15x^2+15xh+5h^2$$
$$f'(x)=15x^2$$

- So far, we have been using $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find the slope of the tangent line of the curve $y = f(x)$ at the general point $(x, f(x))$, also called the derivative, or $f'(x)$ or $\frac{dy}{dx}$.
- This takes a lot of time and mistakes can be easily made if you are not careful. There is an easier way!!

DERIVATIVES : Do you notice the pattern in the following examples?

a) $s(t) = 3t$
$s`(t) = 3$

b) $f(x) = 7x^2$
$f`(x) = 14x$

c) $f(x) = -9x^5$
$f`(x) = -45x^4$

d) $f(x) = \frac{10}{x}$
$f(x) = 10x^{-1}$
$f`(x) = -10x^{-2}$

THE POWER RULE (part 1):

- If $f(x)=x^n$, where n is a real number, then $f'(x)=nx^{n-1}$
- In Leibniz notation we say that $\frac{d}{dx}x^n = nx^{n-1}$.

THE POWER RULE (part 2):

- If $f(x)=cx^n$, where c is a constant and n is a real number, then $f'(x)=(c)(n)x^{n-1}$
- In Leibniz notation we say that $\frac{d}{dx}\left[cf(x)\right]=c\frac{d}{dx}f(x)$

Ex #1: Find $f'(x)$ or $\frac{dy}{dx}$ of the following functions:

a) $f(x)=x^{15}$

b) $f(x)=\frac{1}{x^3}$

c) $f(x)=\sqrt{x}$

d) $f(x)=\frac{1}{\sqrt[3]{x^2}}$

<u>NOTE</u>: At this point we often leave negative exponents in the answers. We will sometimes leave radicals in the denominator and not rationalize the denominator.

Ex #2: Find $f'(x)$ or $\frac{dy}{dx}$ of the following functions:

a) $f(x)=-4x^{15}$

b) $f(x)=(5x^4)^3$

c) $f(x)=\sqrt{7x}$

d) $f(x)=\sqrt[3]{\dfrac{4}{x^2}}$

e) $y=3x^3\sqrt[3]{x}$

THE CONSTANT RULE: If $f(x)=c$, where c is a constant (#), then $f'(x)=0$ (Proof on P 78)

Ex #3: If $f(x)=-5$, determine $f'(x)$.

Ex #4: Find the equation of the tangent line to the curve $y=x^5$ at the point $(2,32)$.

THE SUM/DIFFERENCE RULE:

If $f(x)$ is the sum of 2 differentiable functions $f(x) = g(x) \pm h(x)$) then $f'(x) = g'(x) \pm h'(x)$

Ex #5: Find $f'(x)$ or $\frac{dy}{dx}$ of the following functions:

a) $f(x) = 2x^3 + 7x^6$

b) $f(x) = (4x-3)^2$

c) $f(x) = \frac{\pi x^6}{2} + x - \frac{3}{x}$

d) $f(x) = \frac{(3x-5)(3x+5)}{x^5}$

Ex #6: At what point on the curve $y = -x^2 + 3x + 4$ does the tangent line have a slope of 5?

Ex #7/ A ball is dropped from the upper observation deck of the CN Tower. The distance fallen, in metres, after *t* seconds is $s = 4.9t^2$. How fast is the ball falling at 3 seconds?

Unit 3 : DAY 1 ASSIGNMENT (Section 2.2 & 2.3 in Text)

TEXTBOOK P83 #1, 2a-k, 3ace, 4bc,7,8 and P88 1a-j, 2ac, 3ab , 4 , 7

CALCULUS 30: UNIT 3: DAY 2 – THE PRODUCT RULE (SECTION 2.4)

To learn and apply the PRODUCT rule for differentiation..

When finding the derivative of a function it is definitely easier to use the power rule than it is to use the $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. However, the power will only get us so far. If we were asked to find the derivative of $f(x)=(2x^3-4x^2+x+8)(4x^2+6x-7)$ to use the power rule we would first have to multiply everything else – which could be very time consuming and ugly!

The Product Rule

If $f(x)$ and $g(x)$ are differentiable functions, and $F(x)=f(x)\cdot g(x)$, then $F'(x)=f(x)g'(x)+f'(x)g(x)$
(first times the derivative of the second plus the second times the derivative of the first)

NOTE: It's very important to realize that the derivative of a product DOES NOT equal the product of the derivatives

$$[f(x)g(x)]' \neq f'(x)g'(x)$$

Ex #1: Find the derivative, $\frac{dy}{dx}$ if $y=(2x^3+7)(3x^2-x)$. Use the product law. *(Note: these questions could also be solved by expanding and using the power rule, sum and difference rules .Using the product rule may be more efficient)*

Ex #2: Differentiate $f(x)=\sqrt{x}(2-3x)$ using the product law and simplify. Express your answer using a common denominator.

Ex #3: Find the equation of the tangent line to the graph of $f(x)=(3x^2+2)(2x^3-1)$ when $x=1$.

Ex #4: Find the slope of the tangent line to the function $y=\left(4\sqrt{x}+\frac{2}{x^2}\right)\left(\sqrt[3]{x}-x^3\right)$ at the point x = 1

The pattern using the product rule can continue with more factors:

Ex. $(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$

$(f \cdot g \cdot h \cdot j)' = f' \cdot g \cdot h \cdot j + f \cdot g' \cdot h \cdot j + f \cdot g \cdot h' \cdot j + f \cdot g \cdot h \cdot j'$

And so on . . .

Ex#5/ : If $w(x) = (2x^5)(3x^2 - 4x^{-1})(7x^3 - 6x^{1/2})$ find $w'(x)$. (Don't simplify your answer)(IF TIME PERMITS)

Unit 3: DAY 2 ASSIGNMENT (Section 2.4 in Text)

TEXTBOOK P 92 #1, 2abdeh, 3, 4, 5 , 6 ,9

CALCULUS 30: UNIT 3: DAY 3 – THE QUOTIENT RULE (SECTION 2.5)

To learn and apply the QUOTIENT rule for differentiation..

THE QUOTIENT RULE:

- Given a function in the form of a quotient, $F(x) = \dfrac{f(x)}{g(x)}$, then $F'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

  (Note that we are using a capital F(x) for the quotient function)

- In other words, the derivative of the product of two expressions will be: [(bottom)(derivative of top) – (top)(derivative of bottom)] divided by (bottom squared)
- It is customary NOT to expand the expression in the denominator when applying the quotient rule

Ex #1: Differentiate $F\left(x\right)=\dfrac{x^2+2x-3}{x^3-1}$.

Ex #2: Find $\dfrac{dy}{dx}$ if $y=\dfrac{\sqrt{x}}{1+2x}$.

Ex #3: Find the equation of the tangent line to the curve $f(x)=\dfrac{\sqrt{x}}{x+2}$ at the point (4, -1). Express your answers in GENERAL FORM (Ax + By + C = 0).

Ex #4: Find the coordinates of two points on the graph of the function $f(x)=\dfrac{10x}{x^2+1}$ at which the tangent line is horizontal.

Unit 3: DAY 3 ASSIGNMENT (Section 2.5 in Text)

TEXTBOOK P 95 #1a-l, 2, 3abc, 5, 6 Plus the following

Some ice cubes were added to a cup of boiling water. The temperature of the water in degrees Celsius t minutes after the ice cubes were added, can be approximated by the function

$T(t)=\dfrac{20t^2+100t+200}{t^2+t+2}$. Round your answers to two decimal places where necessary.

(a) Find $T(0)$, $T(1)$, and $T(5)$. Interpret your answers.

(b) Find $T'(t)$.

(c) Find $T'(1)$ and $T'(5)$. Interpret your answers.

(d) Find $\lim\limits_{t\to\infty}\dfrac{20t^2+100t+200}{t^2+t+2}$ and interpret your result.

ANSWER: **21.** (a) $T(0)=100$, $T(1)=80$, $T(5)=37.5$;

initially the water was $100^\circ C$, after 1 minute it was $80^\circ C$, after 5 minutes it was $37.5^\circ C$ (b) $\dfrac{-80t^2-320t}{\left(t^2+t+2\right)^2}$

21. (c) $T'(1)=-25$, $T'(5)=-3.52$; after 1 minute the temperature is falling at a rate of $25^\circ C/\min$ and after 5 minutes the temperature is falling at a rate of $3.52^\circ C/\min$. (d) 20; As time passes the temperature of the water cools towards $20^\circ C$, likely the room temperature. **22.** (a) $L(0)=12000$, $L(5)=112000$,

CALCULUS 30: UNIT 3: DAY 4 – THE CHAIN RULE (SECTION 2.6)

To learn and apply the CHAIN rule for differentiation.

THE CHAIN RULE: (Think back to the composition of functions: Chapter 10 of PC30)

If $F(x)=f(g(x))$, then $F'(x)=f'(g(x))\bullet g'(x)$.

In other words: (derivative of outside function)•(derivative of inside function)

THE POWER RULE COMBINED WITH THE CHAIN RULE:

If $F(x)=\left[f(x)\right]^n$, then $F'(x)=n\left[f(x)\right]^{n-1}\bullet f'(x)$

Reminder: POWER RULE:

If $f(x)=x^n$, $f'(x)=nx^{n-1}$

Ex#1 Find $\frac{dy}{dx}$ if $y=(2x^3-4x+3)^{10}$

Ex #2: a) Differentiate $y=\sqrt[3]{(2x^5-1)^2}$

b) $f\left(x\right)=\frac{1}{\sqrt[3]{1-x^4}}$

Ex #3: If $y = u^{10} + u^5 + 2$ where $u = 1 - 3x^2$, find $\left.\frac{dy}{dx}\right]_{x=1}$ by first using $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Unit 3: DAY 4 ASSIGNMENT (Section 2.6 in Text)

TEXTBOOK P 102 #1a-h, 7

CALCULUS 30:UNIT 3: DAY 5 – THE CHAIN RULE CONT. (SECTION 2.6)

To combine the chain rule with all other rules for differentiation (power, sum, difference, product, quotient).

Ex #1: Differentiate the following. COMPLETE ON LOOSELEAF:

a) $y = (x^2 + 1)^3 (2 - 3x)^4$

b) $s(t) = \left(\frac{2t - 1}{t + 2}\right)^6$

c) $F(x) = \sqrt{x + \sqrt{x^2 + 1}}$

d) y= $\frac{3}{\sqrt{4x - 3}}$

Unit 3: DAY 5 ASSIGNMENT (Section 2.6 in Text):

TEXTBOOK P 102 #1i-l, 6acgil

REVIEW: Converting negative exponents to positive exponents

Ex #1: $\left(\dfrac{2x-1}{x^2+1}\right)^{-5} =$

Ex #2: $\dfrac{(x+1)^{-\frac{2}{3}}}{3x^{-4}(x-2)^2} =$

Ex #3: $\dfrac{1}{2}(x+1)^{-\frac{3}{2}} \cdot 3x^{-3}(2x-3)^2 =$

Factoring (with rational/negative exponents):

- Find the common factor (if there is one)
- Find the smallest exponent
- Remember that when you divide by the common factor, you subtract your exponent.

Ex #4: $2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

Ex #5: $2x^3(x-2)^{-1}(x+1)^{\frac{3}{4}} - 4x^2(x-2)(x+1)^{-\frac{1}{4}}$

Ex #1: Certain functions can be solved using different methods. To find the derivative of the following functions, what rules could you use? Choose the easiest option and differentiate the functions. (COMPLETE ON LOOSELEAF)

a) y= $\dfrac{x^2}{\sqrt{4x-3}}$

b) $f(x) = \dfrac{x(x^2+3)}{(x-2)^4}$

Challenge Question:

c) $f(x) = \dfrac{x^5(x^2+3)^5}{(x-2)^4}$

Find the derivative of each of the following functions, writing your answers in factored form where possible.

1. $f(x)=2x^3+15x^2-36x+12$

2. $f(x)=-2x^{-3}-\frac{1}{2}x^{-2}+x^{-1}+11$

3. $y=\frac{1}{x}+4x$

4. $y=\sqrt{\frac{x}{5}}+\frac{5}{\sqrt{x}}-\frac{x}{\sqrt{5}}$

5. $f(x)=(2x-3)^3(x+1)^2$

6. $f(x)=x^2\sqrt{1-x^2}$

7. $y=(x-2)\sqrt{x^2-3x-1}$

8. $y=4\sqrt{x-1}-6\sqrt{x+1}$

9. $f(x)=\frac{x^2-3x}{x^2+3}$

10. $f(x)=\frac{6}{\sqrt[3]{x^3-2}}$

11. $y=x^3(2x-1)(3x+2)$

12. $y=\frac{x(2x-3)}{x^2+2}$

13. $f(x)=\left(\frac{2x}{x+2}\right)^{-2}$

14. $f(x)=\frac{(x+1)^2}{x^2-2}$

15. $y=\frac{\sqrt{x}}{x^2+1}$

16. $f(x)=\frac{\sqrt{3-x}}{x^4}$

17. $f(x)=\frac{1}{(x^2-2)\sqrt{2x+3}}$

18. $f(x)=\sqrt{\frac{x+4}{x-4}}$

Answers:

1. $6(x-1)(x+6)$ **2.** $-x^{-4}(x-3)(x+2)$ **3.** $x^{-2}(2x-1)(2x+1)$ **4.** $\frac{1}{2\sqrt{5}}x^{-1/2}-\frac{5}{2}x^{-3/2}-\frac{1}{\sqrt{5}}$

5. $10x(x+1)(2x-3)^2$ **6.** $-x(3x^2-2)(1-x^2)^{-1/2}$ **7.** $\frac{1}{2}(4x^2-13x+4)(x^2-3x-1)^{-1/2}$

8. $2(x-1)^{-1/2}-3(x+1)^{-1/2}$ **9.** $\frac{3(x-1)(x+3)}{(x^2+3)^2}$ **10.** $-6x^2(x^3-2)^{-4/3}$ **11.** $2x^2(15x^2+2x-3)$

12. $\frac{3x^2+8x-6}{(x^2+2)^2}$ **13.** $\frac{-(x+2)}{x^3}$ **14.** $\frac{-2(x+1)(x+2)}{(x^2-2)^2}$ **15.** $\frac{1-3x^2}{2x^{1/2}(x^2+1)^2}$ **16.** $\frac{1}{2}x^{-5}(7x-24)(3-x)^{-1/2}$

17. $-(5x^2+6x-2)(x^2-2)^{-2}(2x+3)^{-3/2}$ **18.** $-4(x+4)^{-1/2}(x-4)^{-3/2}$

To learn and apply implicit differentiation versus explicit differentiation.

- So far, our functions have been **explicitly defined**, which is when y is already isolated.
  - Ex: $y = x^5 + 3x - 1$
- We will now be working with **implicitly defined** functions, where we cannot solve for y.
  - Ex: $x^2 - y^3 + 3xy = 1$

Review of Notations:

$\frac{d}{dx}(x^3)$ means: find the derivative of x^3 with respect to x. $\frac{d}{dx}(x^3) =$

$\frac{d}{dx}(y^4)$ means: find the derivative of y^4 with respect to x. $\frac{d}{dx}(y^4) =$ This is challenging as there are no x's

- To take the derivative of a function that is defined implicity, we take the derivaitive from left to right, and wherever there is a value of *y* in the equation, we need to use the chain rule and multiply that term by $\frac{dy}{dx}$. We normally use the chain rule when taking the derivative of x values but the chain rule of those terms ends up being $\frac{dx}{dx}$ which reduces to 1.

Ex #1: Find $\frac{d}{dx}(y^2)$.

Find $\frac{d}{dx}(y^2)$, if $y = (x^2 + 4x + 3)$

Ex #2: Differentiate from left to right with respect to x :

a) $\frac{d}{dx}(9x^2 - 4y^{-\frac{1}{4}})$

b) $\frac{d}{dx}(2x^3y^4)$

USING IMPLICIT DIFFERENTIATION when working with EQUATIONS containing a mixture of x and y:

STEP 1: Differentiate **both sides of the equation**, from left to right, **with respect to x**.

STEP 2: Collect all the terms with $\frac{dy}{dx}$ on one side of the equation

STEP 3: Factor out the $\frac{dy}{dx}$

STEP 4: Isolate $\frac{dy}{dx}$.

Using **implicit differentiation is easier than explicit differentiation**. Here is an example as to why:

Ex #2: If $x^2 + y^2 = 169$, find $\frac{dy}{dx}$ (both explicitly and implicitly).

Ex #3: Suppose $x^2y + 2y^2 - x = 3$.

a) Find $\frac{dy}{dx}$

b) Find the equation of the tangent line at (1,2) in standard form

Ex #4: Find y' if $x^2 + \sqrt{y} = x^2y^3 + 5$

OUTCOME 4B DAY 7 ASSIGNMENT :

TEXTBOOK P 107 #1, 2a-d, f, 3, 5a(Leave answer in standard form)
Challenge yourself with P107 # 6, 7

To learn and apply higher order derivatives.

Higher Order Derivatives:

- We can take the derivative of a derivative function, and the derivative of that function and so on.
- A first derivative is written as $f'(x)$ or $\frac{dy}{dx}$
  - A first derivative represents the slope of a tangent line or rate of change (how the slope of the original function changes). A common example of the first derivative is that velocity is a first derivative of a distance function.
- A second derivative is written as $f''(x)$ or $\frac{d^2y}{dx^2}$
  - A second derivative measures how fast the first derivative function (often velocity) is changing, specifically how the rate of change/slope of the tangent line of the original function changes. A common example of the second derivative is acceleration in that acceleration is the second derivative of a distance function (but the first derivative of a velocity function)
- A third derivative is written as $f'''(x)$ or $\frac{d^3y}{dx^3}$.
  - An example of a third derivative measures how fast acceleration is changing with respect to time. In physics this can also be known as jerk/jolt/surge or lurch.
- If a distance formula y = s(t), then
  y' = v(t) and
  y'' = a(t).
  - If, however, the initial function y = v(t) then it's first derivative y' = a(t)

Ex #1: Find $\frac{d^2y}{dx^2}$ if $y = x^6$

Ex #2: Find the second derivative of $f(x) = 5x^2 + \sqrt{x}$

Ex #3: Find $f''(1)$ if $f(x) = \left(2 - x^2\right)^{10}$

Ex #4: If $x^3 + y^3 = 5$, use implicit differentiation to find $\frac{d^2y}{dx^2}$.

Unit 3 DAY 8 ASSIGNMENT :

TEXTBOOK P 111 #1odd, 2, 3, 4, 5, 7

Unit 3 REVIEW ASSIGNMENT

P 112 #4a-n, 5a, 7abc, 8, 9ade PLUS the following:

1. **Find the coordinates of two points on the graph of $f(x) = 4x^3 + x^2 + 2x + 8$ at which the slope of the tangent line is 4.**
2. **Find $\frac{d^2y}{dx^2}$ given the equation $2y^2 - xy = 6$**

Solutions: 1. $\left(-\frac{1}{2}, \frac{3}{4}\right)$ **and** $\left(\frac{1}{3}, \frac{44}{27}\right)$ **2.** $\frac{d^2y}{dx^2} = \frac{12}{(4y-x)^3}$